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A characterization of coactions which fix Cartan subalgebras

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1 Preparation

In this section, we summarize the basic facts about measured groupoids and von Neumann algebras associated to them. Further details regarding these objects can be found in [3], [8], [9]. We also briefly discuss actions of locally compact quantum groups on von Neumann algebras.

We assume that all von Neumann algebras in this paper have separable preduals, and

$$
\begin{aligned}
(X,\mu) &: \text{standard Borel space},\\
\mathcal{R} &: \text{discrete measured equivalence relation on } (X,\mu),\\
\nu &: \text{left counting measure on } \mathcal{R},\\
\sigma &: \text{normalized 2-cocycle on } \mathcal{R},\\
\mathcal{R}(x) &:= \{y\in X \ : \ (x,y)\in\mathcal{R}\},\\
[\mathcal{R}] &:= \{\varphi : \text{bimeasurable nonsingular transformations}\\
&\qquad\quad \text{such that } \varphi(x) \text{ is in } \mathcal{R}(x) \text{ for a.e. } x \text{ in } X\},\\
\Gamma(\varphi) &:= \{(x,\varphi(x)) : x\in \mathrm{Dom}(\varphi)\} \quad (\varphi\in[\mathcal{R}]).
\end{aligned}
$$

Definition 1. (1) We define a von Neumann algebra $W^*(\mathcal{R},\sigma)$ and a von Neumann subalgebra $W^*(X)$ which act on $L^2(\mathcal{R},\nu)$ by the following:

$$\begin{aligned} W^*(\mathcal{R},\sigma) &:= \{L^\sigma(f) : f \text{ is a left finite function on } \mathcal{R}\}'', \\ W^*(X) &:= \{L^\sigma(d) : d \in L^\infty(X,\mu)\}, \end{aligned}$$

where we regard $L^\infty(X,\mu)$ as functions on the diagonal of $\mathcal{R}$, and $L^\sigma(f)$ is defined by

$$\{L^\sigma(f)\xi\}(x,z) := \sum_{y:(y,x)\in\mathcal{R}} f(x,y)\xi(y,z)\sigma(x,y,z).$$

(2) Let A be a von Neumann algebra and D be a subalgebra of A. We call D is a Cartan subalgebra of A if D satisfies the following:

(i) D is maximal abelian in A,

(ii) D is regular in A, i.e., the normalizer $\mathcal{N}_A(D)$ generates A, where

$$\mathcal{N}_A(D) := \{u \in A : u \text{ is unitary and } uDu^* = D\}.$$

(iii) there exists a faithful normal conditional expectation E_D from A onto D.

Theorem 2 **([3, Theorem 1]).** *For each inclusion of a von Neumann algebra A and a Cartan subalgebra D of A, there exists a standard Borel space (X,μ) and a discrete measured equivalence relation $\mathcal{R}$ on X with a normalized 2-cocycle σ such that $(D \subseteq A)$ is isomorphic to $(W^*(X) \subseteq W^*(\mathcal{R},\sigma))$.*

Theorem 3 **([1, Corollary 3.5]).** *Suppose A is a von Neumann algebra with a Cartan subalgebra D of A such that $A = W^*(\mathcal{R},\sigma)$ and $D = W^*(X)$. Then there exists a bijective correspondence between the set of Borel subrelations $\mathcal{S}$ of $\mathcal{R}$ on (X,μ) and the set of von Neumann subalgebras B of A which contain D:*

$$\begin{aligned} &B \mapsto \mathcal{S}_B \subseteq \mathcal{R} \\ &\mathcal{S} \mapsto W^*(\mathcal{S},\sigma) := \{L^\sigma(f) \in A : \operatorname{supp}(f) \subseteq \mathcal{S}\} \subseteq A. \end{aligned}$$

Let $\mathbb{G} = (M, \Delta, \varphi, \psi)$ be a locally compact quantum group (M is a von Neumann algebra, $\Delta : M \mapsto M \otimes M$ is a coproduct, φ (resp. ψ) is a left (resp. right) invariant weight on M). A normal unital injective $*$-homomorphism α from A onto $M \otimes A$ is called an action of $\mathbb{G}$ on A if α satisfies the following:

$$(\Delta \otimes id_A)\alpha = (id_M \otimes \alpha)\alpha.$$

In particular, if $\mathbb{G}$ is cocommutative, i.e., M is equal to the group von Neumann algebra $W^*(K)$ which is generated by the left regular representation λ_K of a locally compact group K, and Δ is equal to $\hat{\Delta}_K : \lambda_K(k) \mapsto \lambda_K(k) \otimes \lambda_K(k)$, then the action α is called a coaction of K.

2 A reduction to coaction case

In the discussion that follows, we fix a von Neumann algebra A and a Cartan subalgebra D of A with an equivalence relation $\mathcal{R}$ on (X, μ) and a normalized 2-cocycle σ of $\mathcal{R}$ such that the pair $(D \subseteq A)$ is equal to $(W^*(X) \subseteq W^*(\mathcal{R}, \sigma))$.

We assume that the action α fixes D, i.e., $\alpha(d)$ is equal to $1 \otimes d$ for each $d \in D$. It follows that the fixed-point algebra $A^\alpha := \{a \in A : \alpha(a) = 1 \otimes a\}$ is an intermediate subalgebra for $D \subseteq A$.

We will prove that each such a action should be a coaction.

Proposition 4. *Under the situation as above, the von Neumann subalgebra* $\{(id_M \otimes \omega)(\alpha(a)) : a \in A,\ \omega \in A_*\}''$ *of* M *is contained in* $IG(\mathbb{G})''$, *where*

$$IG(\mathbb{G}) := \{u \in M : u \text{ is unitary and } \Delta(u) = u \otimes u\}$$

is the intrinsic group of $\mathbb{G}$.

In particular, if α *is faithful, then* α *is a coaction of some locally compact group.*

Proof. For each $u \in \mathcal{N}_A(D)$, set $w := \alpha(u)(1 \otimes u^*) \in M \otimes A$. Since u normalizes D, for any $d \in D$, we have

$$\begin{aligned} w(1 \otimes d) &= \alpha(u)(1 \otimes u^*)d = \alpha(u)(1 \otimes u^* du)(1 \otimes u^*) \\ &= \alpha(u)\alpha(u^* du)(1 \otimes u^*) = \alpha(du)(1 \otimes u^*) \\ &= (1 \otimes d)w. \end{aligned}$$

Hence w belongs to $(M \otimes A) \cap (\mathbf{C} \otimes D)' = M \otimes D$. So we may and do assume that w is an M-valued function. Moreover, we have

$$\begin{aligned}
(\Delta \otimes id_A)(w) &= (\Delta \otimes id_A)(\alpha(u)(1 \otimes u^*)) \\
&= (\Delta \otimes id_A)(\alpha(u))(1 \otimes 1 \otimes u^*) \\
&= (id_M \otimes \alpha)(\alpha(u))(1 \otimes 1 \otimes u^*) \\
&= (id_M \otimes \alpha)(\alpha(u)(1 \otimes u^*))(1 \otimes \alpha(u))(1 \otimes 1 \otimes u^*) \\
&= w_{12}w_{23}
\end{aligned}$$

Hence w is an $IG(\mathbb{G})$-valued function. So we have that $\alpha(u) = w(1 \otimes u)$ belongs to $IG(\mathbb{G})'' \otimes A$. Since $\mathcal{N}_A(D)$ generates A, we get the conclusion. $\square$

3 Coactions derived from 1-cocycles

Let K be a locally compact group. A Borel map $c : \mathcal{R} \to K$ is called a 1-cocycle if c satisfies the following:

$$\begin{aligned}
c(x,x) &= 1_K \qquad \text{for a.e. } x \in X, \\
c(x,y)c(y,z) &= c(x,z) \qquad \text{for a.e. } (x,y,z) \in \mathcal{R}^2.
\end{aligned}$$

Each 1-cocycle c into K determines a unitary U_c on $L^2(K) \otimes L^2(\mathcal{R})$ by $\{U_c\xi\}(k,x,y) := \xi(c(x,y)^{-1}k,x,y)$. Since c is a 1-cocycle, the map

$$\alpha_c(a) := U_c(1 \otimes a)U_c^* \qquad (a \in A)$$

is a coaction of K. In fact, α_c is defined by the following:

$$\{\alpha_c(L^\sigma(f))\xi\}(k,x,z) := \sum_{y:(y,x)\in\mathcal{R}} f(x,y)\xi(c(x,y)^{-1}k,y,z)\sigma(x,y,z).$$

By the definition of α_c, we have that the fixed-point algebra A^{α_c} is equal to $W^*(\mathrm{Ker}(c),\sigma)$.

We claim that the converse also holds.

Theorem 5. *For each coaction α of K on A which satisfies $D \subseteq A^\alpha \subseteq A$, there exists a Borel 1-cocycle $c : \mathcal{R} \to K$ such that α is equal to α_c.*

Proof. Suppose u is in $\mathcal{N}_A(D)$. By the definition, $\mathrm{Ad}\, u$ determines an automorphism $\rho \in [\mathcal{R}]$. Set $w := \alpha(u)(1 \otimes u^*)$. By using the same argument as in the proof of Proposition 4, w is a $W^*(K)$-valued function. Moreover, for almost all $x \in X$, $w(x)$ is equal to $\lambda_K(k(x))$ for some $k(x) \in K$. We note that the map k depends only on ρ. Now, we define a map c from the graph $\Gamma(\rho^{-1})$ to K by the following:

$$c(\rho(x), x) := k(x) \qquad (x \in \mathrm{Dom}(\rho))$$

By using this construction, we can define a map c from $\mathcal{R}$ to K. We note that the map c is well-defined, i.e., if there exists ρ_1 and ρ_2 in $[\mathcal{R}]$ and a measurable subset $E \subseteq X$ such that $\rho_1(x) = \rho_2(x)$ for all $x \in E$, then there exists null set $F \subseteq X$ such that $c(\rho_1(x), x) = c(\rho_2(x), x)$ for all $x \in E \setminus F$. It is easy to check that c is a 1-cocycle. Moreover, we have that $\alpha(u)$ is equal to $\alpha_c(u)$ for all $u \in \mathcal{N}_A(D)$. Hence we conclude that α is equal to α_c. □

By using the above characterization, we will develop a theory of coactions in terms of 1-cocycles.

In the rest of this paper, we fix a coaction α of K on A and a 1-cocycle $c : \mathcal{R} \to K$ which satisfies $\alpha_c = \alpha$. We denote by $\hat{\mathbb{G}}(K)\,{}_{\alpha_c}\!\ltimes W^*(\mathcal{R}, \sigma)$ the crossed product of A by α, i.e,

$$\hat{\mathbb{G}}(K)\,{}_{\alpha_c}\!\ltimes W^*(\mathcal{R}, \sigma) := (L^\infty(K) \otimes \mathbf{C} \vee \alpha_c(W^*(\mathcal{R}, \sigma))''.$$

We recall that a unitary $V \in W^*(K) \otimes A$ is called an α-1-cocycle if V satisfies the following:

$$(\hat{\Delta}_K \otimes id_A)(V) = V_{23}(id_M \otimes \alpha)(V).$$

Another coaction α' of K on A is said to be cocycle conjugate to α if there exists an α-1-cocycle V and a $*$-automorphism θ of A such that

$$(id_M \otimes \theta) \circ \alpha' \circ \theta^{-1} = \mathrm{Ad}\, V \circ \alpha.$$

For each Borel map $\phi : X \to K$, a unitary $(V_l\xi)(k, x, y) := \xi(\phi(x)^{-1}k, x, y)$ is an α-1-cocycle. So we get the following

Proposition 6. , *Suppose a Borel 1-cocycle $c : \mathcal{R} \to K$ is cohomologous to another Borel 1-cocycle c', i.e., there exists a Borel map $\phi : X \to K$ such that $c'(x, y) = \phi(x)c(x, y)\phi(y)^{-1}$ for a.e. $(x, y) \in \mathcal{R}$. Then the coaction α_c is cocycle conjugate to $\alpha_{c'}$. Hence the crossed product $\hat{\mathbb{G}}(K)\,{}_{\alpha_c}\!\ltimes A$ is isomorphic to $\hat{\mathbb{G}}(K)\,{}_{\alpha_{c'}}\!\ltimes A$.*

4 Connes spectrum and asymptotic range

Let $c : \mathcal{R} \to K$ be a Borel 1-cocycle from an equivalence relation $\mathcal{R}$ into a locally compact group K. Again we consider the coaction α_c of K on the von Neumann algebra $A := W^*(\mathcal{R}, \sigma)$. We will show that the Connes spectrum of the coaction α_c can be described in terms of the 1-cocycle c.

For each such a 1-cocycle $c : \mathcal{R} \to K$, the essential range $\sigma(c)$ is the smallest closed subset F of K such that $c^{-1}(F)$ has complement of ν measure zero. It is easy to check that $k \in K$ belongs to $\sigma(c)$ if and only if, for any (compact) neighborhood U of k, one has $\nu(c^{-1}(U)) > 0$. The asymptotic range $r^*(c)$ of the 1-cocycle c is by definition $\bigcap\{\sigma(c_B) : B \subseteq X,\, \mu(B) > 0\}$, where c_B stands for the restriction of c to the reduction $\mathcal{R}_B$ by B.

Theorem 7. *The Connes spectrum $\Gamma(\alpha_c)$ of α_c is equal to the asymptotic range $r^*(c)$.*

To prove this theorem, we use the following

Lemma 8. *Let $L^\sigma(f) \in A$ and $\omega \in A(K)$, where $A(K)$ is the Fourier algebra $W^*(K)_*$ of K. Then $(\alpha_c)_\omega(L^\sigma(f)) := (\omega \otimes id)(\alpha_c(L^\sigma(f))$ equals $L^\sigma((\omega \circ c)f)$*

Proof. We may and do assume that ω has the form $\omega = \omega_{\eta_1,\eta_2}$ for some η_1, $\eta_2 \in L^2(K)$. For any ζ_1, $\zeta_2 \in L^2(\mathcal{R})$, we have

$$\begin{aligned}
&((\alpha_c)_\omega(L^\sigma(f))\zeta_1 \mid \zeta_2) \\
&= (\alpha_c(L^\sigma(f))(\eta_1 \otimes \zeta_1) \mid \eta_2 \otimes \zeta_2) \\
&= \iint \sum_{y:(y,x)\in\mathcal{R}} \eta_1(c(x,y)^{-1}k)\overline{\eta_2(k)} \cdot f(x,y)\zeta_1(y,z)\sigma(x,y,z)\overline{\zeta_2(x,z)}d\nu(x,z)dk \\
&= \int \sum_{y:(y,x)\in\mathcal{R}} \omega(c(x,y))f(x,y)\zeta_1(y,z)\sigma(x,y,z)\overline{\zeta_2(x,z)}d\nu(x,z) \\
&= (L^\sigma((\omega \circ c)f)\zeta_1 \mid \zeta_2).
\end{aligned}$$

Thus we are done. □

Proof of Theorem 7. Since the center $\mathcal{Z}(A^\alpha)$ is contained in D, we have

$$\Gamma(\alpha_c) = \bigcap\{\mathrm{Sp}((\alpha_c)^e) : e \ : \text{non-zero projection in } D\}.$$

Hence, it suffices to show that $\mathrm{Sp}(\alpha_c) = \sigma(c)$.

Let $k \in \sigma(c)$. Take any compact neighborhood U of k. Since $\nu(c^{-1}(U)) > 0$, there exists a measurable subset $E \subseteq c^{-1}(U)$ such that $\nu(E) > 0$ and $L^\sigma(\chi_E) \in A$. Then define $a := L^\sigma(\chi_E) \in A \setminus \{0\}$. If $\omega \in A(K)$ vanishes on some neighborhood of U, then, by Lemma 8, we have $(\alpha_c)_\omega(a) = 0$. From [6, Chapter IV, Lemma 1.2 (ii)], it follows that $\mathrm{Sp}_{\alpha_c}(a) \subseteq U$. Hence a belongs to $A^{\alpha_c}(U)$. By [6, Chapter IV, Lemma 1.2 (iv)], k lies in $\mathrm{Sp}(\alpha_c)$.

Conversely suppose that $k \in \mathrm{Sp}(\alpha_c)$. We will show that, for each open neighborhood V of k, $c^{-1}(V)$ is not a ν-null set. Indeed, if $\nu(c^{-1}(V))$ is equal to 0 for some V, we have $L^\sigma(f) = L^\sigma(f\chi_{c^{-1}(V)^c})$ for each $L^\sigma(f) \in A$. So, for each $\omega \in A(K)$ such that $\mathrm{supp}\,\omega \subseteq U$, by Lemma 8, we have

$$(\alpha_c)_\omega(L^\sigma(f)) = L^\sigma(f\chi_{c^{-1}(V)^c}(\omega \circ c)) = 0.$$

So we conclude that $(\alpha_c)_\omega(a) = 0$ for each $a \in A$ and $\omega \in A(K)$ such that $\mathrm{supp}\,\omega \subseteq U$. In the meantime, since V is open, for each $h \in V$, there exists $\omega \in A(K)$ such that $\omega(h) = 1$ and $\mathrm{supp}\,\omega \subseteq V$. This shows that for each $a \in A$, $h \notin \mathrm{Sp}_{\alpha_c}(a)$. This contradicts [6, Chapter IV, Lemma 1.2(iv)]. Therefore k belongs to $\sigma(c)$. $\square$

By using the above theorem and [4, Lemma 1.13], we get the following

Corollary 9 (cf. [5]). *Let A be an AFD type* II *factor. Suppose that α and α' are coactions of a locally compact group K on A such that each of A^α and $A^{\alpha'}$ contains a Cartan subalgebra of A. If $\Gamma(\alpha) = \Gamma(\alpha') = K$, then α is cocycle conjugate to α'.*

Proof. Suppose that A^α (resp. $A^{\alpha'}$) contains a Cartan subalgebra D_1 (resp. D_2) of A. By [2], there exists a $*$-automorphism θ of A such that $\theta(D_1) = D_2$. Set $\alpha_\theta := (id_{W^*(K)} \otimes \theta^{-1}) \circ \alpha \circ \theta$. Then we have $A^{\alpha_\theta} = \theta(A^\alpha)$. So $D_2 = \theta(D_1) \subseteq \theta(A^\alpha) = A^{\alpha_\theta}$. Clearly, α_θ is cocycle conjugate to α. Hence it suffices to assume from the outset that $D_1 = D_2 =: D$.

We may assume that the inclusion $(D \subseteq A)$ is of the form $(L^\infty(X) \subseteq W^*(\mathcal{R}))$ for an amenable ergodic type II equivalence relation $\mathcal{R}$ on a standard Borel space (X, μ) with an invariant measure μ. By Theorem 5 there exist Borel 1-cocycles c and c' from $\mathcal{R}$ to K such that $\alpha = \alpha_c$ and $\alpha' = \alpha_{c'}$. By Theorem 7, we have $r^*(c) = r^*(c') = K$. So we may apply [4, Lemma 1.13], and obtain that there exist cocycles $\overline{c}$ and $\overline{c'}$ cohomologous to c and c' respectively as 1-cocycles on $\mathcal{R}$ such that $\overline{c}$ is equal to $\overline{c'} \circ \rho$ for some $\rho \in N[\mathcal{R}]$, the normalizer of $\mathcal{R}$. By Proposition 6, α (resp. α') is cocycle conjugate to $\alpha_{\bar{c}}$ (resp. $\alpha_{\bar{c'}}$). Furthermore, a direct computation shows that for each $X \in W^*(\mathcal{R})$,

$$\alpha_{\bar{c}\circ\rho}(X) = (1 \otimes \Phi_\rho^{-1})(\alpha_{\bar{c}}(\Phi_\rho(X))),$$

where Φ_ρ is an automorphism on $W^*(\mathcal{R})$ which is defined by

$$\Phi_\rho(L(f)) := L(f \circ \rho).$$

So we conclude that $(1 \otimes \Phi_\rho)\alpha_{\bar{c}\circ\rho} = \alpha_{\bar{c}} \circ \Phi_\rho$, i.e., $\alpha_{\bar{c}\circ\rho}$ is conjugate to $\alpha_{\bar{c}}$. Hence α is cocycle conjugate to α'. □

5 Exchangeability for a 1-cocycle with a smaller range within the cohomology class

Suppose that there exists a closed subgroup H of K which cohomologous to c and the range is contained in H. By regarding c' as a 1-cocycle into H, we obtain the crossed product $\hat{\mathbb{G}}(H)_{\alpha_{c'}} \ltimes A$ and the dual action $\widehat{\alpha_{c'}}$ of H. It follows that the dual action $\widehat{\alpha_c}$ of K is induced from $\widehat{\alpha_{c'}}$. Namely, there exists an isomorphism Π from $\hat{\mathbb{G}}(K)_{\alpha_c} \ltimes A$ onto $L^\infty(K/H) \otimes (\hat{\mathbb{G}}(H)_{\alpha_{c'}} \ltimes A)$ such that $\Pi \circ \widehat{(\alpha_c)}_k = \delta_k \circ \Pi$, where the action δ of K is the induced action of $\widehat{\alpha_{c'}}$([7]).

We will show that the converse also holds.

Theorem 10 (cf. [9, Theorem 3.5]). *Let $c : \mathcal{R} \to K$ be a Borel 1-cocycle and H be a closed subgroup of K. Then the following are equivalent:*

(1) There exists a Borel 1-cocycle $c_0 : \mathcal{R} \to K$, cohomologous to c, such that the range of c_0 is contained in H.

(2) There exists an injective $*$-homomorphism Θ from $L^\infty(K/H)$ into the center of the crossed product $\widehat{\mathbb{G}}(K)_{\alpha_c} \ltimes A$ such that $\Theta \circ \ell_k = \widehat{(\alpha_c)}_k \circ \Theta$ for all $k \in K$, where ℓ_k comes from the left translation by k on K/H. Equivalently, if Y is the measure-theoretic spectrum of the center of the crossed product (i.e., the measure space on which the Mackey action (the Poincaré flow) of K is considered), then it is an extension of the K-space K/H.

(3) The covariant system $\{\widehat{\mathbb{G}}(K)_{\alpha_c} \ltimes A,\ K,\ \widehat{\alpha_c}\}$ is induced from some system $\{P,\ H,\ \beta\}$.

If one of (1) $\sim$ (3) occurs, then one can take $\{P, H, \beta\}$ to be $\{\widehat{\mathbb{G}}(H)_{\alpha_{c'}} \ltimes A, H, \widehat{\alpha_{c'}}\}$, where $c' : \mathcal{R} \to H$ is the 1-cocycle obtained by regarding c_0 as an H-valued 1-cocycle.

Proof. It is easy to check hat the condition (2) follows (1). By using the Imprimitivity Theorem of [7], (2) is equivalent to (3). So we will prove (2)$\Rightarrow$(1).

If such a $*$-homomorphism Θ exists, then by using [7], the dual action $\widehat{(\alpha_c)}_k$ is induced from an action β of H on a von Neumann algebra P. We denote the induced action of β by δ. By the assumption, there exists a $*$-isomorphism Π from $\widehat{\mathbb{G}}(K)_{\alpha_c} \ltimes A$ onto $L^\infty(K/H) \otimes P$ such that $\Pi \circ \widehat{(\alpha_c)}_k = \delta_k \circ \Pi$ for all $k \in K$.

A direct computation shows that $\Pi(\alpha_c(A))$ is equal to $\mathbf{C} \otimes P^\beta$. Moreover, since β is defined by $\beta_h := \mathrm{Ad}(\lambda_H(h) \otimes 1)|_P$, there exists a dual action β' on H which is conjugate to β. So there exist a von Neumann algebra B and a coaction τ of H on B such that the dual action $\widehat{(\alpha_c)}$ is conjugate to the induced action by $\hat{\tau}$. In particular, we have

$$\widehat{\mathbb{G}}(K)_{\alpha_c} \ltimes A \cong L^\infty(K/H) \otimes \widehat{\mathbb{G}}(H)_\tau \ltimes B$$

Under the above isomorphism, we have that there exists a isomorphism η from A onto B such that the fixed-point subalgebra B^τ contains a Cartan subalgebra $\eta(D)$. So τ comes from a 1-cocycle $c_0 : \mathcal{R} \to H$. By the construction, we conclude that c_0 is cohomologous to c as a cocycle into K.

Therefore we complete the proof. $\square$

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